COMPLEX REPRESENTATION GROWTH OF FINITE QUASISIMPLE GROUPS OF LIE TYPE

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Abstract. We give upper bounds to the number of n-dimensional irreducible complex representations of finite quasisimple groups belonging to different families of groups of Lie type. The bounds have the form cn^s, where c and s are explicit positive constants that both depend on the family in question. From these bounds, we deduce a uniform bound of the form cn to the number of n-dimensional irreducible representations of all finite quasisimple groups of Lie type. Finally, an application of these results to counting conjugacy classes of maximal subgroups of Lie groups is discussed.

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1. Introduction

Let $r_n(G)$ denote the number of inequivalent irreducible n-dimensional complex linear representations of a group G. Letting n grow indefinitely, we are interested in bounding the growth of $r_n(G)$ from above, when G is a finite quasisimple group with $G/Z(G)$ a simple group of Lie type.

Martin Liebeck and Aner Shalev have proved in [12] the following upper bound for the growth rate: if $\mathcal{L}$ is a fixed Lie type and $\mathcal{L}(q)$ denotes any finite quasisimple group of type $\mathcal{L}$ over the field $\mathbb{F}_q$, then

$$r_n(\mathcal{L}(q)) < c_{\mathcal{L}}\, n^{2/h_{\mathcal{L}}} \qquad \text{for all } n. \tag{1}$$

Here $h_{\mathcal{L}}$ is the Coxeter number of $\mathcal{L}$ (number of roots divided by Lie rank), and $c_{\mathcal{L}}$ is some constant that only depends on the Lie type. The bound is asymptotically tight in the sense that the exponent cannot be made smaller. Liebeck and Shalev used their result to find mixing times for random walks on groups of Lie type.

In this paper, we present numerical upper bounds for the total number of irreducible n-dimensional complex representations of groups that appear in certain families $\mathcal{L}$ of finite quasisimple groups of Lie type. In other words, letting $r_n^f(G)$ denote the number of faithful irreducible n-dimensional representations of G, we will bound the sum

$$s_n(\mathcal{L}) = \sum_{G\in\mathcal{L}} r_n^f(G)$$

for various classes $\mathcal{L}$. In fact, the bounds are of the following type:

$$s_n(\mathcal{L}) < cn^s, \tag{2}$$

where c and s are constants that only depend on the family $\mathcal{L}$. The results are presented in Theorems 1–3.

We have elsewhere obtained similar numerical bounds for quasisimple groups where the simple quotient is an alternating group (see [8]). Combining these with the results presented here, it is possible to estimate the number of conjugacy classes of certain type of closed proper subgroups of classical Lie groups, namely those that normalise a finite quasisimple group. This was already done by R. Guralnick, M. Larsen and C. Manack in their paper [5], using earlier results by the author. These results were without explicit numerical constants and were communicated privately by the author and his supervisor, Prof. Martin Liebeck. We shall briefly explain the application in the final section of this paper.

1.1. **The results.** We first deal with the classical groups. We concern ourselves with the following families:

A_1 :	linear groups in dimension 2
A' :	linear groups in dimension at least 3
^2A :	unitary groups in dimension at least 3
B :	orthogonal groups in odd dimension ≥ 7 over a field of odd size
C :	symplectic groups in dimension at least 4
D :	orthogonal groups of plus type in even dimension ≥ 8
^2D :	orthogonal groups of minus type in even dimension ≥ 8.

Regarding that the Liebeck–Shalev bound given by (1) is asymptotically tight, we realise that for each $\mathcal{L}$, the smallest possible value for the exponent s in (2) is $2/h_{\mathcal{L}}$, where $h_{\mathcal{L}}$ is the largest Coxeter number appearing in the family $\mathcal{L}$. As a matter of fact, we can often push the exponent down to $2/h_{\mathcal{L}}$, but not always. (For details, see the remark on page 24.) Beside these optimal exponents, we list our best efforts as $s_{\mathcal{L}}$ in Table 1. Excluding type A_1, we will consider exponents 1, 2/3 and $s_{\mathcal{L}}$ for each classical family $\mathcal{L}$.

$\mathcal{L}$:	A_1	A'	^2A	B	C	D	^2D
$2/h_{\mathcal{L}}$:	1	2/3	2/3	1/3	1/2	1/3	1/3
$s_{\mathcal{L}}$:	1	2/3	2/3	**1/2**	1/2	**1/2**	**1/2**

TABLE 1. Optimal and obtained bounding exponents for the classical families. Where the optimal exponent differs from the obtained one, the latter is written in boldface.

Theorem 1. *For all $n > 1$, we have*

$$s_n(A_1) \leq \tfrac{8}{3}n.$$

Also, for $n > 12$, we have

$$s_n(A_1) \leq n.$$

Remark. The constant in the first bound in the previous theorem is smallest possible, since there are eight 3-dimensional representations of quasisimple groups of type A_1. The second bound is also tight, which we came to realise only recently. The bound appears in a slightly weaker form in [8] and [7].

Theorem 2. *Let $\mathcal{L}$ denote one of the families of classical groups presented above, and let $s_{\mathcal{L}}$ be as in Table 1. Then, for s in $\{1, 2/3, s_{\mathcal{L}}\}$, we have*

$$s_n(\mathcal{L}) \leq c_{\mathcal{L},s}\, n^s \qquad \text{for all } n > 1,$$

with the constants $c_{\mathcal{L},s}$ given in Table 2.

s	A_1	A'	${}^2A$	B	C	D	${}^2D$
1	8/3	7/8	2/3	2/27	1/2	1/8	1/34
2/3	—	7/4	3.464	2/9	0.8144	0.2804	0.1681
$s_{\mathcal{L}}$	8/3	7/4	3.464	0.3850	1.3417	2.134	2.134

Table 2. Bounding constants for classical groups.

Remark. For the previous theorem, the exact values of $s_n(\mathcal{L})/n^s$ were computed for small values of n. The maximal values of the ratio are given in Table 20 on page 29. If an entry in that table equals the corresponding one in Table 2, then we know that in that case the discovered constant is smallest possible (at least for small n).

Next, let $\mathcal{E}$ denote the family of finite quasisimple groups of exceptional Lie types ${}^2B_2$, ${}^3D_4$, E_6, E_7, E_8, ${}^2E_6$, F_4, ${}^2F_4$, G_2 and ${}^2G_2$.

Theorem 3. *For all $n > 1$, we have*

(a) $s_n(\mathcal{E}) \leq n$
(b) $s_n(\mathcal{E}) \leq 1.913n^{2/3}$
(c) $s_n(\mathcal{E}) < 5.057n^{1/2}$.

Remark. The multiplicative constants in parts (a) and (b) of the above theorem are smallest possible, for there are altogether seven 7-dimensional representations of quasisimple groups of types G_2 and ${}^2G_2$. Also, the exponent 1/2 is smallest possible, since each group ${}^2B_2(q)$ has $q/2-1$ representations of dimension q^2+1.

We also get the immediate corollary below.

Corollary 4. *Let $\mathcal{H}$ denote the class of all finite quasisimple groups of Lie type. We have*

$$s_n(\mathcal{H}) \leq \tfrac{5}{3}n \qquad \text{for all } n > 12.$$

For $n \leq 12$, the values of $s_n(\mathcal{H})$ are given in Table 3.

Proof. While deriving the main results, we will have computed the multiplicities of all representation degrees less than 2000 for all quasisimple groups of Lie type. The maximum of the ratio $s_n(\mathcal{H})/n$ is reached at $n = 24$, and it equals 5/3, as there are altogether 40 representations of dimension 24. When $n \geq 2000$, we use the bounds in Theorems 1, 2 and 3 to see that

$n:$	2	3	4	5	6	7	8	9	10	11	12
$s_n:$	2	10	12	9	25	20	19	13	17	5	23

TABLE 3. Values of $s_n(\mathcal{H})$ for $n < 13$.

the stated upper bound holds. Specifically, we use the bound n for A_1, the bottom line of Table 2 for the other classical types, and the bound in part c) of Theorem 3 for exceptional types. □

1.2. **Overview and structure of the paper.** We obtain the above results by examining minimal character degrees of groups of Lie type. For all exceptional groups and for classical groups of small rank, the complete information of character degrees is available in tables produced by F. Lübeck and available on his website [13]. For classical groups of large rank, however, we shall have to do with so-called *gap results*, which give the few smallest character degrees.

The main bulk of the work has to do with classifying certain polynomials giving the character degrees in such a way that they can then be managed computationally. The origin of these polynomials is explained in the preliminary Section 2. The classification process is then given in Section 3, first for polynomials arising from Lübeck's data (Subsection 3.1) and then for those arising from gap results (Subsection 3.2). The actual results are proved in Section 4 for the classical groups (Theorems 1 and 2) and in Section 5 for the exceptional groups (Theorem 3). In the final Section 6, we explain the application of our results in the study of conjugacy classes of maximal subgroups of classical Lie groups.

2. Preliminaries

A universal covering group of a finite simple group of Lie type is completely determined by its simple quotient, which in turn is determined by its Lie family, rank r and the size q of the defining field. We will almost exclusively deal with the universal covering groups, as every irreducible representation of a quasisimple group can be lifted to a representation of the universal covering group. For this reason, we shall denote the universal covering group simply by $H_r(q)$, where H is replaced by the letter of the Lie family in question. For example, $A_2(3)$ is the group $\mathrm{SL}_3(3)$, and $E_6(4)$ is the triple cover of the finite simple group of Lie type E_6 over the field of four elements. Notice that this notation differs from the usual one, as $E_6(4)$, for example, is here not the simple group but rather its universal cover. The notation is also different from the one used in the ATLAS of Finite Groups [2].

Apart from 19 exceptions, the universal covering groups of simple groups of Lie type are obtained as follows. Suppose $\mathbf{G}(\bar{\mathbb{F}}_q)$ is a simply-connected simple linear algebraic group over the algebraic closure of $\mathbb{F}_q$, and let F be a Frobenius morphism. In the general case, the finite fixed point group $\mathbf{G}(q) = \mathbf{G}(\bar{\mathbb{F}}_q)^F$ is a universal covering group. We call these covering groups *regular*. The exceptions to this rule are given e.g. in [9, Theorem 5.1.4], and they are: $A_1(4)$, $A_1(9)$, $A_2(2)$, $A_2(4)$, $A_3(2)$, ${}^2A_3(2)$, ${}^2A_3(3)$, ${}^2A_5(2)$, $B_3(3)$, $C_2(2)$, $C_3(2)$, $D_4(2)$, ${}^2E_6(2)$, $F_4(2)$, $G_2(2)$, $G_2(3)$, $G_2(4)$, ${}^2B_2(8)$ and ${}^2F_4(2)$ (the Tits group). Even in these exceptional cases, the universal covering group is always denoted $H_r(q)$.

Let us now concentrate on the regular universal covering groups and see how their character degrees can be obtained. The behaviour of irreducible characters of finite groups of Lie type can be understood by the means of Deligne–Lusztig theory. A brief account of the theory is given in [3, Chapters 11–13]. We are mainly interested in the partitioning of characters into so-called *Lusztig series*, and in how the degrees can be calculated from the partition. To describe this, we need the concept of dual algebraic groups.

Let $\mathbf{G}$ and $\mathbf{G}^*$ be simple algebraic groups over an algebraically closed field $\bar{\mathbb{F}}_q$, with maximal tori T and T^*, respectively. The group $\mathbf{G}^*$ is said to be *dual* to $\mathbf{G}$ if there is an isomorphism between the character group $X(T)$ and the cocharacter group $Y(T^*)$ sending roots to coroots. In particular, the dual group to a simply-connected simple algebraic group is always an adjoint group. For example, the groups $\mathrm{SL}_n(\bar{\mathbb{F}}_q)$ and $\mathrm{PGL}_n(\bar{\mathbb{F}}_q)$ are dual to each other. If the groups are also endowed with Frobenius morphisms F and F^*, these morphisms are said to be dual to each other if they are compatible with the above isomorphism. More information about duality can be found in [1, Chapter 4].

Now, by Deligne–Lusztig theory, the characters of $\mathbf{G}^F$, a finite group of Lie type, are partitioned into Lusztig series ([3, Proposition 13.17]). Each Lusztig series $\mathcal{E}(\mathbf{G}^F, (s))$ is labelled by a so-called *geometric conjugacy class* (s) of a semi-simple element s in the dual group $(\mathbf{G}^*)^{F^*}$, and the members of $\mathcal{E}(\mathbf{G}^F, (1))$ are called *unipotent characters*. For any semi-simple $s \in (\mathbf{G}^*)^{F^*}$, there is a bijection ψ_s between $\mathcal{E}(\mathbf{G}^F, (s))$ and $\mathcal{E}(C_{\mathbf{G}^*}(s)^{F^*}, (1))$, the unipotent characters of the centraliser of s in the dual group ([3, Theorem 13.23]). Moreover, there is the following formula for the degree of a character χ in $\mathcal{E}(\mathbf{G}^F, (s))$ (see [3, Remark 13.24]):

$$\chi(1) = \frac{\left|\mathbf{G}^F\right|_{p'}}{\left|C_{\mathbf{G}^*}(s)^{F^*}\right|_{p'}}\, \psi_s(\chi)(1). \tag{3}$$

Here, p is the characteristic of $\mathbb{F}_q$, and subscript p' denotes the p-prime part.

The unipotent characters of adjoint groups were discovered by Lusztig, and they are exhibited in [1, Chapters 13.8–13.9]. Their degrees are given by polynomials in q (except for the Suzuki and Ree types ${}^2B_2$, ${}^2F_4$ and ${}^2G_2$, for which the polynomials are in terms of $\sqrt{q}$). Let now $H_r(q)$ be a simply-connected finite group of a fixed Lie type H_r. It follows from the formula (3), together with the exact form of the degrees of the unipotent characters of adjoint groups, as well as the formulae for the orders of finite groups of Lie type (see e.g. [1, page 75]), that the character degrees of $H_r(q)$ are given by a finite set of polynomials in q (or $\sqrt{q}$ for the Suzuki and Ree groups). These sets of degree polynomials may differ depending on the congruence class of q modulo a fixed number that depends only on the type H_r. The degree polynomials are henceforth referred to as *Lusztig polynomials*.

We shall make use of the Lusztig polynomials in two ways. For Lie types of rank at most 8 (in some cases 7), Frank Lübeck has produced listings of the Lusztig polynomials. He has also computed the multiplicities of these polynomials, that is, how many characters share the same degree polynomial. The lists are available at his website [13]. The multiplicities are also polynomials in q. (A description of how these lists were produced is given in [15].) On the other hand, P. Tiep and A. Zalesskii [18] have found the polynomials giving the few smallest character degrees of classical groups, regardless of rank, and this list has been extended by Guralnick and Tiep [6] and H. N. Nguyen [17] for the symplectic and orthogonal groups.

Looking at equation (3) more closely, it is evident that the resulting Lusztig polynomials are products of cyclotomic polynomials and a monomial (with a possible positive rational constant). The following lemma gives some simple facts about these kinds of polynomials.

Lemma 5. *Consider a polynomial $f(x)$ that is a product of a power of x and a non-zero number of cyclotomic polynomials, not all of them $x+1$ nor all of them $x-1$. Suppose that*

$$f(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_1x + a_0.$$

Let k be the greatest integer smaller than d such that $a_k \neq 0$, and write $a_M = \max_{i<k-1} |a_i|$. Let also x_0 be the larger root of $|a_k|x^2 - (|a_k| + |a_{k-1}|)x - (a_M - |a_{k-1}|)$. Then the following are true.

(a) The function $x \mapsto f(x)$ is strictly increasing for all $x > 1$.
(b) Either

$$x^d < f(x) < (x+1)^d \tag{4}$$

or

$$(x-1)^d < f(x) < x^d \tag{5}$$

holds for all $x > x_0$. Moreover, case (4) holds if and only if $a_k > 0$.

Proof. (a) Note that f has the general form

$$f(x) = \prod_{i=1}^{d} (x - \omega_i),$$

where each ω_i is either zero or a root of unity, and not all of them are 0, 1 nor -1. When $x > 1$, the modulus $|x - \omega_i|$ of each factor is positive and increasing in x, and thus their product $|f(x)|$ is strictly increasing in x. Since the value $f(x)$ is real and positive for $x > 1$, it follows that part (a) holds.

(b) Assume $x > 1$. Now, as $|x-1| \leq |x - \omega_i| \leq |x+1|$ is true for every i in the above product, and equality does not hold on either side for all i, we get

$$|x-1|^d < |f(x)| < |x+1|^d.$$

For $x > 1$, all parts of this inequality are real and positive, so we may drop the absolute value signs. It remains to show that $f(x) - x^d$ does not pass over zero when $x > x_0$.

Let $x > x_0$. Assume first that $a_k > 0$. Then

$$f(x) - x^d = a_k x^k + a_{k-1}x^{k-1} + \sum_{i=0}^{k-2} a_i x^i \geq a_k x^k - |a_{k-1}|x^{k-1} - a_M \sum_{i=0}^{k-2} x^i$$
$$= a_k x^k - |a_{k-1}|x^{k-1} - a_M \frac{x^{k-1}-1}{x-1} \geq x^{k-1}\left(a_k x - |a_{k-1}| - \frac{a_M}{x-1}\right).$$

Multiplying through by $x-1$, it is straightforward to see that when $x > x_0$, the last expression is greater than 0. Similarly, if $a_k < 0$, we get

$$x^d - f(x) = |a_k|x^k - a_{k-1}x^{k-1} - \sum_{i=0}^{k-2} a_i x^i \geq |a_k|x^k - |a_{k-1}|x^{k-1} - a_M \sum_{i=0}^{k-2} x^i$$
$$= |a_k|x^k - |a_{k-1}|x^{k-1} - a_M \frac{x^{k-1}-1}{x-1} \geq x^{k-1}\left(|a_k|x - |a_{k-1}| - \frac{a_M}{x-1}\right).$$

Again, the last expression is greater than zero when $x > x_0$. □

3. Classifying the degree polynomials

In this section, we explain how to classify and parametrise the Lusztig polynomials in order to perform computations on them. We focus our attention to the universal covering groups of classical groups of rank greater than 1, that is, groups that belong to one of the following families mentioned in the Introduction: A', ${}^2A$, B, C, D and ${}^2D$. Moreover, we will only deal with universal covering groups of regular type, as the exceptional covers can be taken care of one by one.

There are systematic isomorphisms between certain small-rank groups in different families (e.g. between B_2 and C_2), and we wish to take each group into account only once. We have therefore excluded some of the smallest ranks in some of the families. For example, for family B we take the smallest applicable rank to be 3, since $B_1(q) \cong A_1(q)$ and $B_2(q) \cong C_2(q)$ for every q. The smallest applicable ranks are denoted as $r_0 = r_0(\mathcal{L})$ and listed in Table 4. Also, we will not consider groups of type B in even characteristic, as $B_r(2^k) \cong C_r(2^k)$ for all r.

The complete lists of Lusztig polynomials are available for groups of small rank. We will write $r_1 = r_1(\mathcal{L})$ for the first rank for which Lübeck has not produced complete lists with multiplicities. In other words, lists are available for $r < r_1$. The values of r_1 are also given in Table 4.

$\mathcal{L}$:	A'	${}^2A$	B	C	D	${}^2D$
r_0 :	2	2	3	2	4	4
r_1 :	9	9	9	9	8	8

Table 4. Restrictions for rank in the classical families.

3.1. **Classifying Lübeck's polynomials.** Fix any classical family $\mathcal{L}$ and restrict attention to groups in this family with rank at least $r_0(\mathcal{L})$ and less than $r_1(\mathcal{L})$. Then the Lusztig polynomials of these groups, together with their multiplicities, appear in Lübeck's lists. We also assume that $q \geq q_1 = 49$, for the smaller groups will be dealt with by direct computation.

We shall describe a list of pairs of polynomials (f_i, h_i), labelled by some index set I, where the f_i will run through all Lusztig polynomials pertaining to the family $\mathcal{L}$ (and having suitable rank). Thus, each $f_i(q)$ will be a representation degree of some universal covering group $H_r(q)$ of type $\mathcal{L}$, with $r_0 \leq r < r_1$. (As $q \geq q_1$, the universal covering group will be of regular type.) On the other hand, the h_i will be chosen so that, for each i, the value $h_i(q)$ will be an upper bound for the multiplicity of the representation degree $f_i(q)$. We achieve this enumeration in the following steps 1–4.

Step 1. Fix a rank r with $r_0 \leq r < r_1$, and collect into one set F_r all degree polynomials of the groups $H_r(q)$. Enumerate the polynomials as $f_{r,i}$, with i in some index set I_r.

Step 2. For each pair (r, i), with $i \in I_r$, there may exist several multiplicity polynomials $g_1, \ldots, g_k$ for the degree polynomial $f_{r,i}$. (These correspond to different congruence classes of q.) Going through the g_j one by one, construct a polynomial $g_{r,i}$ by the following algorithm:

(a) Let $g_{r,i,1} = g_1$.
(b) For any $j > 1$, if one of $g_j(q)$ and $g_{r,i,j-1}(q)$ dominates the other for all $q \geq q_1$, write $g_{r,i,j}$ for the dominating one. Otherwise, let $g_{r,i,j} = g_j + g_{r,i,j-1}$.
(c) Let $g_{r,i} = g_{r,i,k}$.

This procedure ensures that $g_{r,i}(q)$ is always an upper bound for the multiplicity of $f_{r,i}(q)$.

Step 3. Let $F = \bigcup_r F_r$ and enumerate the new set as $F = \{f_i\}_{i \in I}$. For all $i \in I$, define

$$\eta_i = \sum_{f_{r,i} = f_i} g_{r,i}.$$

In other words, η_i is the sum of all those $g_{r,i}$ that correspond to a given f_i as the latter appears as $f_{r,i}$ for different ranks r.

Step 4. Lastly, to make certain calculations easier, we define h_i to be the polynomial obtained from η_i by disregarding all terms with negative sign.

We now have all the Lusztig polynomials appearing in Lübeck's lists indexed by I. Next, we partition the indices $i \in I$ with regard to the asymptotic behaviour of f_i. More precisely, for any positive rational number a and integer d, we define the following parametrised classes:

$$I^+_{a,d} = \{i \in I \mid aq^d \leq f_i(q) < a(q+1)^d \text{ for } q \geq q_1\}$$
$$I^-_{a,d} = \{i \in I \mid a(q-1)^d < f_i(q) < aq^d \text{ for } q \geq q_1\}.$$

All these sets are clearly disjoint. By going through all Lübeck's lists, we have checked that no degree polynomial has the form $a(q-1)^d$ or $a(q+1)^d$. Now, Lemma 5 ensures that for large enough q_1, each $i \in I$ is contained in one of $I^{\pm}_{a,d}$. Again, we have checked computationally that the q_1 we have chosen is adequate. Thus we know that the sets $I^{\pm}_{a,d}$ form a partition of I.

For the sake of convenience, we shall often write I_ν instead of $I^{\varepsilon}_{a,d}$, abbreviating the parameters as one triple $\nu = (a, d, \varepsilon)$. When ε appears in formulae, we will also write 0 for $+$ and 1 for $-$.

To each triple $\nu = (a, d, \varepsilon)$ we further attach a positive integer N_ν and a polynomial h_ν. Firstly, we let N_ν be the smallest value given by the degree polynomials indexed by I_ν for $q \geq q_1$:

$$N_\nu = \min_{i \in I_\nu} f_i(q_1).$$

Secondly, the polynomial h_ν is defined as follows. For most parameter vectors ν, it can be computationally verified that there is some $i \in I_\nu$ for which h_i dominates the other multiplicities, that is, $h_i(q) \geq \max_{j \in I_\nu} h_j(q)$ for all $q \geq q_1$. In this case, we define $h_\nu = h_i$. Otherwise, we let $h_\nu = \sum_{i \in I_\nu} h_i$.

Example 6. The Lusztig polynomials of the groups $A_2(q)$ are given in the two tables below. The degree polynomials are listed as φ_i, and the multiplicity polynomials as ψ_i. The first table is for q congruent to 0 or 2 modulo 3, and the second is for q congruent to 1 modulo 3.

We see that some degree polynomials appear in both tables. We enumerate the degree polynomials as $f_{2,i}$. Of those multiplicity polynomials that correspond to the same degree polynomial in different tables, we choose the bigger one and call it $g_{2,i}$. For example, polynomials φ_3 and φ_{12} are equal, so they will be taken as one polynomial $f_{2,3}$, and the chosen multiplicity will be $g_{2,3}(q) = \frac{1}{3}(q^2 + q)$.

i	φ_i	ψ_i
1	$q^2 + q$	1
2	$q^2 + q + 1$	$q - 2$
3	$q^3 - q^2 - q + 1$	$\frac{1}{3}(q^2 + q)$
4	$q^3 - 1$	$\frac{1}{2}(q^2 - q)$
5	q^3	1
6	$q^3 + q^2 + q$	$q - 2$
7	$q^3 + 2q^2 + 2q + 1$	$\frac{1}{6}(q^2 - 5q + 6)$

i	φ_i	ψ_i
8	$q^2 + q$	1
9	$q^2 + q + 1$	$q - 2$
10	$\frac{1}{3}(q^3 - q^2 - q + 1)$	6
11	$\frac{1}{3}(q^3 + 2q^2 + 2q + 1)$	3
12	$q^3 - q^2 - q + 1$	$\frac{1}{3}(q^2 + q - 2)$
13	$q^3 - 1$	$\frac{1}{2}(q^2 - q)$
14	q^3	1
15	$q^3 + q^2 + q$	$q - 2$
16	$q^3 + 2q^2 + 2q + 1$	$\frac{1}{6}(q^2 - 5q + 4)$

According to the previous discussion, the next step would be to combine the tables for different ranks. We pretend there is only one rank to be considered, so that $f_i = f_{2,i}$ for all i and the polynomials h_i will be obtained from the $g_{2,i}$ simply by disregarding any negative terms. (In reality, we would take sums of different $g_{r,i}$.) This leads to the following table:

i	$f_{2,i} = f_i$	$g_{2,i}$	h_i	$\nu = (a, d, \varepsilon)$
1	$q^2 + q$	1	1	$(1, 2, +)$
2	$q^2 + q + 1$	$q - 2$	q	
3	$\frac{1}{3}(q^3 - q^2 - q + 1)$	6	6	$(1/3, 3, -)$
4	$\frac{1}{3}(q^3 + 2q^2 + 2q + 1)$	3	3	$(1/3, 3, +)$
5	$q^3 - q^2 - q + 1$	$\frac{1}{3}(q^2 + q)$	$\frac{1}{3}(q^2 + q)$	$(1, 3, -)$
6	$q^3 - 1$	$\frac{1}{2}(q^2 - q)$	$\frac{1}{2}q^2$	
7	q^3	1	1	$(1, 3, +)$
8	$q^3 + q^2 + q$	$q - 2$	q	
9	$q^3 + 2q^2 + 2q + 1$	$\frac{1}{6}(q^2 - 5q + 6)$	$\frac{1}{6}(q^2 + 6)$	

In the above table, we have also partitioned the rows according to the classes I_ν, whose parameters are displayed in the last column. As an example, for the class with $\nu = (1, 3, -)$ we would have $N_\nu = f_5(q_1) = 115200$ and $h_\nu = h_6$.

The significance of the classification described above is revealed in the following lemma.

Lemma 7. *Let $n > 1$. Writing $H_r(q)$ for a regular universal covering group of a fixed classical family, we have*

$$\sum_{r_0 \le r < r_1} \sum_{q \ge q_1} r_n\big(H_r(q)\big) \;\le\; \sum_{\nu;\, N_\nu \le n} h_\nu\big((n/a)^{1/d} + \varepsilon\big),$$

where $\nu = (a, d, \varepsilon)$ and $\varepsilon \in \{0, 1\}$.

Proof. The idea is to estimate the double sum by adding together all multiplicities of degree polynomials that obtain a particular value n.

Suppose $n > 1$ is a representation degree of some $H_r(q)$ with $r_0 \le r < r_1$ and $q \ge q_1$. Such a degree is given by $f_i(q)$ for some indices i. We argue that in any I_ν there is at most one such i.

Firstly, if $f_i(q) = n$ for some $i \in I^+_{a,d}$, then we have $aq^d \le n < a(q + 1)^d$, and this inequality holds for at most one integer q. The conclusion is also true if $i \in I^-_{a,d}$. On the other hand, we can check computationally that when $q \ge q_1$, we have $f_i(q) \ne f_j(q)$ for any i and j in I_ν, unless $i = j$. Thus, there is at most one $i \in I_\nu$ for which $f_i(q) = n$.

Finally, the polynomial f_i indexed by I_ν cannot have n as a value unless $N_\nu \le n$. On the other hand, if $n = f_i(q)$ and $i \in I_\nu$, we have $q < (n/a)^{1/d} + \varepsilon$. As h_ν is an upper bound for all the h_i with $i \in I_\nu$, the result holds. $\square$

3.2. **Minimal degree polynomials and gap results.** For ranks at least $r_1(\mathcal{L})$, we do not have complete lists of Lusztig polynomials available. Instead, we will use partial results obtained by Tiep and Zalesskii [18], Nguyen [17] and Guralnick and Tiep [6]. These results give us some of the Lusztig polynomials corresponding to the smallest representation degrees.

The minimal degree results are sometimes called *gap results* because they show that there are only a few characters of the smallest degree n_1, and maybe a few of the degrees n_1+1 and n_1+2, whereafter there is a relatively large "gap" before the next degree. Again, after a few degrees, there might be another gap before the next one and so forth.

For example, any group $A_r(q) = \mathrm{SL}_{r+1}(q)$ with $r \geq 2$ has irreducible representations of degrees

$$\begin{aligned} \frac{q^{r+1}-q}{q-1} &= q^r + q^{r-1} + \cdots + q \\ \text{and} \quad \frac{q^{r+1}-1}{q-1} &= q^r + q^{r-1} + \cdots + q + 1. \end{aligned}$$

The difference between these two degrees is one, and they are both "below the first gap". The next dimension is "above the first gap", given by a polynomial with degree at least $2r-2$.

For each classical family $\mathcal{L}$, we will now describe a similar classification of the minimal degree polynomials as we did above with Lübeck's polynomials. For $r \geq r_1(\mathcal{L})$, we have listed the necessary polynomials in Tables 5 through 10. In the tables, φ_i always denotes the polynomial giving the i'th smallest character degree, and ψ_i is the corresponding multiplicity. Notice that the multiplicities are linear polynomials, and do not depend on the rank of the group like the degrees do. In some cases the polynomials depend on the congruence class of q, and this is indicated in a separate column.

Fix a family $\mathcal{L}$. We shall classify the set of minimal degree polynomials of groups of type $\mathcal{L}$ according to their asymptotic behaviour. If φ_i is such a minimal degree polynomial, let a be its leading coefficient and $k(r)$ its degree. Then it is straightforward to check that, for each i, exactly one of the following holds:

$$aq^{k(r)} < \varphi_i(r,q) < a(q+1)^{k(r)} \qquad \text{for all } q \text{ and } r \tag{6}$$

$$\text{or} \qquad a(q-1)^{k(r)} < \varphi_i(r,q) < aq^{k(r)} \qquad \text{for all } q \text{ and } r. \tag{7}$$

The degree of the polynomial has the form $k(r) = \alpha r + \beta$, where α and β are integers. We classify all the minimal degree polynomials according to properties (6) and (7) by partitioning the indices i into classes $I^{\varepsilon}_{a,\alpha,\beta}$, where ε is set to $+$ if φ_i satisfies condition (6), and to $-$ otherwise. The parameters are shown next to the corresponding polynomials in the tables. We shall often abbreviate the parameters as one vector $\nu = (a, \varepsilon, \alpha, \beta)$, writing I_ν instead of $I^{\varepsilon}_{a,\alpha,\beta}$. In formulae, we shall also write ε as 0 instead of $+$ and 1 instead of $-$.

The last row in the tables is a lower bound to the next character degree, so it indicates the size of the next gap. We call the polynomial appearing in this row the *gap bound*, and denote it by $\Gamma = \Gamma_{\mathcal{L}}$. Degrees given by the actual minimal degree polynomials are said to be *below the gap bound*, and the bigger degrees are *above the gap bound*.

For each gap bound, we can also find a lower bound of the form

$$\Gamma(r,q) \geq a(q-\varepsilon)^{\alpha r+\beta},$$

where a is the leading coefficient of Γ, ε is either 0 or 1, and α and β are integers. Thus, we may attach a parameter vector $(a,\varepsilon,\alpha,\beta)$ also to the gap bound, although this polynomial is not considered to be indexed by any I_ν. The parameters are shown in the tables.

	$\varphi_i(r,q)$	$\psi_i(q)$	$(a,\varepsilon,\alpha,\beta)$
	$\frac{q^{r+1}-q}{q-1}$	1	$(1,+,1,0)$
	$\frac{q^{r+1}-1}{q-1}$	$q-2$	
Γ:	$\frac{(q^{r+1}-1)(q^r-q^2)}{(q-1)(q^2-1)}$	—	$(1,+,2,-2)$

Table 5. Minimal degree polynomials of $A_r(q)$, $r \geq 9$. (From [18].)

	$\varphi_i(r,q)$	$\psi_i(q)$	q	$(a,\varepsilon,\alpha,\beta)$
	$\frac{q^{r+1}-q+\kappa(q-1)}{q+1}$	$1+\kappa(q-1)$		$(1,-,1,0)$
	$\frac{q^{r+1}+1+\kappa(q-1)}{q+1}$	$q+\kappa(1-q)$		
Γ:	$\frac{(q^{r+1}+1-2\kappa)(q^r-q^2+\kappa(q^2-q))}{(q+1)(q^2-1)}$	—	2	$(1,-,2,-2)$
	$\frac{(q^{r+1}+1-2\kappa)(q^r-q^2+\kappa(q^2+1))}{(q+1)(q^2-1)}$	—	>2	

Table 6. Minimal degree polynomials of ${}^2A_r(q)$, $r \geq 9$. Here, $\kappa = 0$ if r is even, and $\kappa = 1$ if r is odd. (From [18].)

	$\varphi_i(r,q)$	$\psi_i(q)$	q	$(a,\varepsilon,\alpha,\beta)$
	$\frac{q^{2r}-1}{q^2-1}$	1	odd	$(1,+,2,-2)$
	$\frac{(q^r-1)(q^r-q)}{2(q+1)}$	1		$(1/2,-,2,-1)$
	$\frac{(q^r+1)(q^r+q)}{2(q+1)}$	1		
	$\frac{(q^r+1)(q^r-q)}{2(q-1)}$	1		$(1/2,+,2,-1)$
	$\frac{(q^r-1)(q^r+q)}{2(q-1)}$	1		
	$\frac{q^{2r}-1}{q+1}$	$\frac{1}{2}q$	even	$(1,-,2,-1)$
	$\frac{q^{2r}-1}{q+1}$	$\frac{1}{2}(q-1)$	odd	
	$\frac{q^{2r}-1}{q-1}$	$\frac{1}{2}(q-2)$	even	$(1,+,2,-1)$
	$\frac{q^{2r}-1}{q-1}$	$\frac{1}{2}(q-3)$	odd	
	$\frac{q^{2r+1}-q}{q^2-1}$	1	odd	
Γ:	q^{4r-8}	—		$(1,+,4,-8)$

TABLE 7. Minimal degree polynomials of $B_r(q)$, $r \geq 9$. (From [17] for odd q and [6] for even q.)

$\varphi_i(r,q)$	$\psi_i(q)$	q	$(a,\varepsilon,\alpha,\beta)$
$\frac{1}{2}(q^r-1)$	2	odd	$(1/2,-,1,0)$
$\frac{1}{2}(q^r+1)$	2	odd	$(1/2,+,1,0)$
$\frac{(q^r-1)(q^r-q)}{2(q+1)}$	1		$(1/2,-,2,-1)$
$\frac{q^{2r}-1}{2(q+1)}$	2	odd	
$\frac{(q^r+1)(q^r+q)}{2(q+1)}$	1		
$\frac{(q^r+1)(q^r-q)}{2(q-1)}$	1		$(1/2,+,2,-1)$
$\frac{q^{2r}-1}{2(q-1)}$	2	odd	
$\frac{(q^r-1)(q^r+q)}{2(q-1)}$	1		
$\frac{q^{2r}-1}{q+1}$	$\frac{1}{2}q$	even	$(1,-,2,-1)$
$\frac{q^{2r}-1}{q+1}$	$\frac{1}{2}(q-1)$	odd	
$\frac{q^{2r}-1}{q-1}$	$\frac{1}{2}(q-2)$	even	$(1,+,2,-1)$
$\frac{q^{2r}-1}{q-1}$	$\frac{1}{2}(q-3)$	odd	
$\frac{(q^{2r}-1)(q^{r-1}-q)}{2(q^2-1)}$	2	odd	$(1/2,+,3,-3)$
$\frac{(q^{2r}-1)(q^{r-1}-1)}{2(q^2-1)}$	2	odd	
$\frac{(q^{2r}-1)(q^{r-1}+1)}{2(q^2-1)}$	2	odd	
$\frac{(q^{2r}-1)(q^{r-1}+q)}{2(q^2-1)}$	2	odd	
$\frac{(q^{2r}-1)(q^{r-1}-1)}{2(q+1)}$	$q-1$	odd	$(1/2,-,3,-2)$
$\frac{(q^{2r}-1)(q^{r-1}+1)}{2(q+1)}$	$q-1$	odd	
$\frac{(q^{2r}-1)(q^r-q)}{2(q^2-1)}$	2	odd	$(1/2,+,3,-2)$
$\frac{(q^{2r}-q^2)(q^r-1)}{2(q^2-1)}$	2	odd	
$\frac{(q^{2r}-q^2)(q^r+1)}{2(q^2-1)}$	2	odd	
$\frac{(q^{2r}-1)(q^r+q)}{2(q^2-1)}$	2	odd	
$\frac{(q^{2r}-1)(q^{r-1}-1)}{2(q-1)}$	$q-3$	odd	
$\frac{(q^{2r}+1)(q^{r-1}-1)}{2(q-1)}$	$q-3$	odd	
Γ: $\left(\frac{(q^{r-1}+1)(q^{r-2}-q)}{q^2-1}-1\right)\frac{q^{r-1}(q^{r-1}-1)(q-1)}{2}$	—	even	$(1/2,-,4,-6)$
$\frac{(q^{2r-1}-1)(q^{r-1}-1)(q^{r-1}-q^2)}{2(q^4-1)}$	—	odd	

TABLE 8. Minimal degree polynomials of $C_r(q)$, $r \geq r_0$. (From [17] for odd q and [6] for even q.)

	$\varphi_i(r,q)$	$\psi_i(q)$	q	$(a,\varepsilon,\alpha,\beta)$
	$\frac{(q^r-1)(q^{r-1}+q)}{q^2-1}$	1		$(1,+,2,-3)$
	$\frac{(q^r-1)(q^{r-1}-1)}{2(q+1)}$	2	odd	$(1/2,-,2,-2)$
	$\frac{(q^r-1)(q^{r-1}+1)}{q-1}$	2	odd	$(1/2,+,2,-2)$
	$\frac{(q^r-1)(q^{r-1}-1)}{q+1}$	$\frac{1}{2}q$	even	$(1,-,2,-2)$
	$\frac{(q^r-1)(q^{r-1}-1)}{q+1}$	$\frac{1}{2}(q-1)$	odd	
	$\frac{q^{2r}-q^2}{q^2-1}$	1		$(1,+,2,-2)$
	$\frac{(q^r-1)(q^{r-1}+1)}{q-1}$	$\frac{1}{2}(q-2)$	even	
	$\frac{(q^r-1)(q^{r-1}+1)}{q-1}$	$\frac{1}{2}(q-3)$	odd	
Γ:	$q^{4r-10}+1$	—		$(1,+,4,-10)$

TABLE 9. Minimal degree polynomials of $D_r(q)$, $r \geq 8$. (From [17].)

	$\varphi_i(r,q)$	$\psi_i(q)$	q	$(a,\varepsilon,\alpha,\beta)$
	$\frac{(q^r+1)(q^{r-1}-q)}{q^2-1}$	1		$(1,+,2,-3)$
	$\frac{(q^r+1)(q^{r-1}+1)}{2(q+1)}$	2	odd	$(1/2,-,2,-2)$
	$\frac{(q^r+1)(q^{r-1}-1)}{q-1}$	2	odd	$(1/2,+,2,-2)$
	$\frac{(q^r+1)(q^{r-1}+1)}{q+1}$	$\frac{1}{2}q$	even	$(1,-,2,-2)$
	$\frac{(q^r+1)(q^{r-1}+1)}{q+1}$	$\frac{1}{2}(q-1)$	odd	
	$\frac{q^{2r}-q^2}{q^2-1}$	1		$(1,+,2,-2)$
	$\frac{(q^r+1)(q^{r-1}-1)}{q-1}$	$\frac{1}{2}(q-2)$	even	
	$\frac{(q^r+1)(q^{r-1}-1)}{q-1}$	$\frac{1}{2}(q-3)$	odd	
Γ:	$q^{4r-10}+1$	—		$(1,+,4,-10)$

TABLE 10. Minimal degree polynomials of ${}^2D_r(q)$, $r \geq 8$. (From [17].)

We add still some more properties to the classes of minimal degree polynomials constructed above. To each parameter vector $\nu = (a, \varepsilon, \alpha, \beta)$ we attach (as was done earlier with Lübeck's polynomials) certain bounding numbers and polynomials. Firstly, for some ν as indicated in Tables 5–10, there is a limitation for those field sizes to which the polynomials indexed by I_ν can be applied. We write $q_0(\nu)$ for the smallest such field. Note that $q_0(\nu)$ is either 2 or 3, depending on ν, and that for type B we have $q_0(\nu) = 3$ for all ν.

Secondly, define

$$f_\nu(r, q) = a(q - \varepsilon)q^{\alpha r + \beta - 1}. \tag{8}$$

(In the formula, we write $\varepsilon \in \{0, 1\}$.) For each ν, it can be checked that f_ν is a lower bound for the degree polynomials indexed by I_ν. More precisely, for all $i \in I_\nu$ and each $r \geq r_1(\mathcal{L})$ and $q \geq q_0(\nu)$, we have $\varphi_i(r, q) \geq f_\nu(r, q)$. Note that if $\varepsilon = +$, then $f_\nu(r, q) = aq^{\alpha r + \beta}$.

For each ν, there is a multiplicity polynomial ψ_i with $i \in I_\nu$ that dominates all other multiplicity polynomials in the same class. We let g_ν denote this largest multiplicity polynomial, in addition dropping any terms with negative sign. Thus, $g_\nu(q) \geq \psi_i(q)$ for all $i \in I_\nu$. Note that all the g_ν are first degree polynomials.

Lastly, we let $N_\nu(r)$ be the smallest value attained by the minimal degree polynomials indexed by I_ν and corresponding to a group of rank $r \geq r_1(\mathcal{L})$. The value of $N_\nu(r)$ is obtained by substituting $q_0(\nu)$ for q in the smallest minimal degree polynomial indexed by I_ν.

The values of $q_0(\nu)$, g_ν and $N_\nu(r)$ are gathered together in Tables 11 through 16. In these tables, the last row gives bounds related to the gap bound. For a parameter vector μ corresponding to a gap bound Γ, we let f_μ denote a polynomial lower bound to Γ with the same form as in (8). The value of $q_0(\mu)$ is also defined in the same way as $q_0(\nu)$ above.

	$\nu = (a, \varepsilon, \alpha, \beta)$	$q_0(\nu)$	$g_\nu(q)$	$N_\nu(r)$
	$(1, +, 1, 0)$	2	q	$2^{r+1} - 2$
gap:	$(1, +, 2, -2)$	2	—	—

Table 11. Bounding parameters for minimal degree polynomials of $A_r(q)$.

	$\nu = (a, \varepsilon, \alpha, \beta)$	$q_0(\nu)$	$g_\nu(q)$	$N_\nu(r)$
	$(1, -, 1, 0)$	2	q	$\bigl(2^{r+1} - \gcd(r, 2)\bigr)/3$
gap:	$(1, -, 2, -2)$	2	—	—

Table 12. Bounding parameters for minimal degree polynomials of ${}^2A_r(q)$.

	$\nu = (a, \varepsilon, \alpha, \beta)$	$q_0(\nu)$	$g_\nu(q)$	$N_\nu(r)$
	$(1, +, 2, -2)$	3	1	$(3^{2r} - 1)/8$
	$(1/2, -, 2, -1)$	3	1	$(3^{2r} - 4 \cdot 3^r + 3)/8$
	$(1/2, +, 2, -1)$	3	1	$(3^{2r} - 2 \cdot 3^r - 3)/4$
	$(1, -, 2, -1)$	3	$q/2$	$(3^{2r} - 1)/2$
	$(1, +, 2, -1)$	3	$q/2$	$(3^{2r} - 1)/2$
gap:	$(1, +, 4, -8)$	3	—	—

Table 13. Bounding parameters for minimal degree polynomials of $B_r(q)$.

	$\nu = (a, \varepsilon, \alpha, \beta)$	$q_0(\nu)$	$g_\nu(q)$	$N_\nu(r)$
	$(1/2, -, 1, 0)$	3	2	$(3^r - 1)/2$
	$(1/2, +, 1, 0)$	3	2	$(3^r + 1)/2$
	$(1/2, -, 2, -1)$	2	2	$(2^{2r-1} - 3 \cdot 2^{r-1} + 1)/3$
	$(1/2, +, 2, -1)$	2	2	$2^{2r-1} - 2^{r-1} - 1$
	$(1, -, 2, -1)$	2	$q/2$	$(2^{2r} - 1)/3$
	$(1, +, 2, -1)$	2	$q/2$	$2^{2r} - 1$
	$(1/2, +, 3, -3)$	3	2	$(3^{3r-1} - 3^{2r+1} - 3^{r-1} + 3)/16$
	$(1/2, -, 3, -2)$	3	q	$(3^{3r-1} - 3^{2r} - 3^{r-1} + 1)/8$
	$(1/2, +, 3, -2)$	3	q	$(3^{3r} - 3^{2r+1} - 3^r + 3)/16$
gap:	$(1/2, -, 4, -6)$	2	—	—

Table 14. Bounding parameters for minimal degree polynomials of $C_r(q)$.

	$\nu = (a, \varepsilon, \alpha, \beta)$	$q_0(\nu)$	$g_\nu(q)$	$N_\nu(r)$
	$(1, +, 2, -3)$	2	1	$(2^{2r-1} + 3 \cdot 2^{r-1} - 1)/3$
	$(1/2, -, 2, -2)$	3	2	$(3^{2r-1} - 4 \cdot 3^{r-1} + 1)/8$
	$(1/2, +, 2, -2)$	3	2	$(3^{2r-1} + 2 \cdot 3^{r-1} - 1)/2$
	$(1, -, 2, -2)$	2	$q/2$	$(2^{2r-1} - 3 \cdot 2^{r-1} + 1)/3$
	$(1, +, 2, -2)$	2	$q/2$	$(2^{2r} - 4)/3$
gap:	$(1, +, 4, -10)$	2	—	—

Table 15. Bounding parameters for minimal degree polynomials of $D_r(q)$.

	$\nu = (a, \varepsilon, \alpha, \beta)$	$q_0(\nu)$	$g_\nu(q)$	$N_\nu(r)$
	$(1, +, 2, -3)$	2	1	$(2^{2r-1} - 3 \cdot 2^{r-1} - 2)/3$
	$(1/2, -, 2, -2)$	3	2	$(3^{2r-1} + 4 \cdot 3^{r-1} + 1)/8$
	$(1/2, +, 2, -2)$	3	2	$(3^{2r-1} - 2 \cdot 3^{r-1} - 1)/2$
	$(1, -, 2, -2)$	2	$q/2$	$(2^{2r-1} + 3 \cdot 2^{r-1} + 1)/3$
	$(1, +, 2, -2)$	2	$q/2$	$(2^{2r} - 4)/3$
gap:	$(1, +, 4, -10)$	2	—	—

TABLE 16. Bounding parameters for minimal degree polynomials of ${}^2D_r(q)$.

From the parameter vectors ν, we can always reconstruct the bounding polynomials f_ν and f_μ defined by (8). For example, for the family D, these lower bounds read as follows:

ν	f_ν
$(1,+,2,-3)$	q^{2r-3}
$(1/2,-,2,-2)$	$\frac{1}{2}(q-1)q^{2r-3}$
$(1/2,+,2,-2)$	$\frac{1}{2}q^{2r-2}$
$(1,-,2,-2)$	$(q-1)q^{2r-3}$
$(1,+,2,-2)$	q^{2r-2}
$\mu=(1,+,4,-10)$	q^{4r-10}

Let now $H_r(q)$ be a universal covering group belonging to a classical family $\mathcal{L}$. Consider a parameter vector ν. For any fixed positive integer n, the lower bound given in (8) yields upper bounds to those field sizes q and ranks r for which n can appear as one of the irreducible representation degrees of $H_r(q)$ indexed by I_ν. Namely, we see that if $\varphi_i(r,q)=n$ for some $i\in I_\nu$, then

$$r \le \frac{1}{\alpha}\left(\frac{\log n-\log a-\varepsilon\log(q_0-1)}{\log q_0}-\beta+\varepsilon\right), \tag{9}$$

where $q_0=q_0(\nu)$. We write $r_\nu(n)$ for this upper bound. Note especially that $N_\nu(r)>n$ for $r>r_\nu(n)$.

Also, for each r we have

$$q \le \left(\frac{n}{a(q_0-1)^\varepsilon}\right)^{\frac{1}{\alpha r+\beta-\varepsilon}}. \tag{10}$$

Call this upper bound $q_\nu(r,n)$. Similar bounds can also be obtained if the degree n is above the gap bound. These will be denoted r_μ and q_μ for the gap parameter vector μ.

As an example, we give in Tables 17 and 18 the expressions for r_ν, q_ν, r_μ and q_μ in the case $\mathcal{L}=D$, and also the expressions for q_μ for all classical families.

ν	$q_0(\nu)$	$r_\nu(n)$	$q_\nu(r,n)$
$(1,+,2,-3)$	2	$\frac{1}{2}(\log_2 n+3)$	$n^{\frac{1}{2r-3}}$
$(1/2,-,2,-2)$	3	$\frac{1}{2}(\log_3 n+3)$	$n^{\frac{1}{2r-3}}$
$(1/2,+,2,-2)$	3	$\frac{1}{2}(\log_3 n+\log_3 2+2)$	$(2n)^{\frac{1}{2r-2}}$
$(1,-,2,-2)$	2	$\frac{1}{2}(\log_2 n+3)$	$n^{\frac{1}{2r-3}}$
$(1,+,2,-2)$	2	$\frac{1}{2}(\log_2 n+2)$	$n^{\frac{1}{2r-2}}$
$\mu=(1,+,4,-10)$	2	$\frac{1}{4}(\log_2 n+10)$	$n^{\frac{1}{4r-10}}$

Table 17. Expressions for r_ν, q_ν, r_μ and q_μ for the Lie family D.

$\mathcal{L}:$	A'	${}^2A$	B	C	D	${}^2D$
$q_\mu(r,n):$	$n^{\frac{1}{2r-2}}$	$n^{\frac{1}{2r-3}}$	$n^{\frac{1}{4r-8}}$	$(2n)^{\frac{1}{4r-7}}$	$n^{\frac{1}{4r-10}}$	$n^{\frac{1}{4r-10}}$

TABLE 18. Expressions for q_μ for all classical families.

These bounds can now be used in the same way as the corresponding bounds for Lübeck's polynomials.

Lemma 8. *Write $H(q)$ for a universal covering group of a fixed classical type $\mathcal{L}$ with fixed rank $r \geq r_1(\mathcal{L})$, and let $r_n^<(H(q))$ be the number of irreducible representations of degree n below the gap bound. Then we have*

$$\sum_q r_n^<\big(H(q)\big) \;\leq \sum_{\nu;\, N_\nu(r)\leq n} g_\nu\big(q_\nu(r,n)\big).$$

Proof. The proof is similar to the one of Lemma 7. Suppose $n > 1$ is a representation degree of some $H(q)$. Such a degree is given by a minimal degree polynomial φ_i. We argue that in any I_ν there is at most one such i.

As in the proof of Lemma 7, it follows from the definition of the classes I_ν that there is only one possible q for which $\varphi_i(q) = n$ can hold. On the other hand, by going through all minimal degree polynomials of type $\mathcal{L}$, we can check that they all have distinct values at any q (when $r \geq r_1(\mathcal{L})$). Hence, there can exist at most one i such that $\varphi_i(q) = n$.

Now, a minimal degree polynomial giving the degree n cannot be indexed by I_ν unless $N_\nu(r) \leq n$. In this case, $q_\nu(r,n)$ is an upper bound to the possible q for which $H(q)$ can have n as a representation degree, and g_ν dominates all multiplicity polynomials related to I_ν. This justifies the upper bound. □

3.3. **Above the gap bound.** Assume that $H_r(q)$ is a universal covering group of a classical type $\mathcal{L}$ and $r \geq r_1(\mathcal{L})$, as before. If a representation degree of $H_r(q)$ is above the gap bound, we do not know its exact multiplicity. In this case, we shall simply bound the multiplicity by the number of conjugacy classes of $H_r(q)$. Upper bounds to these conjugacy class numbers have been obtained by J. Fulman and R. Guralnick in [4]. The bounds have the form $q^r + B_{\mathcal{L}}q^{r-1}$, where each $B_{\mathcal{L}}$ is a constant depending on the classical family $\mathcal{L}$. The constants are listed in Table 19.

$\mathcal{L}:$	A'	${}^2A$	B	C	D	${}^2D$
$B_{\mathcal{L}}:$	3	15	22	30	32	32

TABLE 19. Constants in the Fulman–Guralnick bounds for conjugacy class numbers of classical groups.

The following two simple lemmata allow us to bound the multiplicities of the representation degrees in a fairly effective way. The first one is a simple estimate relieving us from having to sum over all integers when we cannot determine which of them are prime powers.

Lemma 9. *Suppose that K is a strictly increasing function on the integers and Q is a positive integer. For summing the values of $K(q)$ over prime powers q, we have the following estimate:*

$$\sum_{q\leq Q} K(q) \ < \ K(Q) + \frac{1}{2}\sum_{i=3}^{Q-1} K(i) + \sum_{i=1}^{[\log_2 Q]} K(2^i).$$

(Here $[x]$ denotes the integral part of x.)

Proof. Firstly, each q can be either odd or a power of two. The binary powers are handled by the second sum on the right hand side.

For odd values of q, we have two possibilities. If Q is even, the integers from 3 to Q can be partitioned into pairs $(2l-1, 2l)$, and we have $K(2l) > K(2l-1)$ for all l, since K is strictly increasing. Thus,

$$\sum_{\substack{q\leq Q\\ q \text{ odd}}} K(q) \leq \sum_{l=2}^{Q/2} K(2l-1) < \frac{1}{2}\sum_{l=2}^{Q/2}\big(K(2l-1)+K(2l)\big) = \frac{1}{2}\sum_{i=3}^{Q} K(i).$$

On the other hand, if Q is odd, we have by the same argument

$$\sum_{\substack{q\leq Q\\ q \text{ odd}}} K(q) < K(Q) + \frac{1}{2}\sum_{i=3}^{Q-1} K(i).$$

In both cases, we see that the desired inequality holds. □

Lemma 10. *Keep the notation of the previous lemma. Additionally, assume that $K(q)$ has the form $q^r + Bq^{r-1}$ for some $B > 0$ and $r \geq 2$. For the summation of $K(q)$ over prime powers q, we obtain*

$$\sum_{q\leq Q} K(q) < \frac{1}{2(r+1)}Q^{r+1} + \left(\frac{2^{r+1}-1}{2^r-1} + \frac{B}{2r}\right)Q^r + \frac{(2^r-1)B}{2^{r-1}-1}Q^{r-1}.$$

Proof. The result follows directly from the lemma above. Notice first that

$$\frac{1}{2}\sum_{i=3}^{Q-1} K(i) < \frac{1}{2}\int_0^Q K(x)\,\mathrm{d}x = \frac{Q^{r+1}}{2(r+1)} + \frac{BQ^r}{2r}. \tag{11}$$

Also, using the formula for geometric sum, we get

$$\begin{aligned}\sum_{i=1}^{[\log_2 Q]} K(2^i) &= \sum_{i=1}^{[\log_2 Q]}\left(2^{ri} + B\cdot 2^{(r-1)i}\right)\\ &= \frac{2^r\left(2^{[\log_2 Q]r}-1\right)}{2^r-1} + \frac{B\cdot 2^{r-1}\left(2^{[\log_2 Q](r-1)}-1\right)}{2^{r-1}-1}\\ &< \frac{2^rQ^r}{2^r-1} + \frac{2^{r-1}BQ^{r-1}}{2^{r-1}-1}. \end{aligned}\tag{12}$$

Putting together (11) and (12), and adding $K(Q)$ yields the result. □

4. Proving the main results on classical groups

Let $\mathcal{L}$ stand for one of the classical families A', ${}^2A$, B, C, D or ${}^2D$, and let $H_r(q)$ denote a universal covering group of type $\mathcal{L}$, having rank $r \geq r_0$ and being defined over a field of size q. We shall use the parametrisations of Lusztig polynomials described in the previous section to compute upper bounds to

$$s_n(\mathcal{L}) = \sum_r \sum_q r_n\big(H_r(q)\big).[1]$$

These upper bounds will then furnish the results stated in Theorems 1 and 2. We start with groups of rank greater than 1, postponing the case $\mathcal{L} = A_1$.

Proof of Theorem 2, excluding the case $\mathcal{L} = A_1$. Fix the value of s among $\{1, 2/3, s_{\mathcal{L}}\}$. (The values for $s_{\mathcal{L}}$ were given in Table 1.) We strive to find uniform bounds for the ratios

$$Q_n(\mathcal{L}, s) = \frac{1}{n^s} \sum_r \sum_q r_n\big(H_r(q)\big).$$

We begin by giving a description of different kinds of bounds we shall need, and after this deal with each family $\mathcal{L}$ separately. For the general discussion, assume that the universal covering group $H_r(q)$ is of regular type.

Let us first consider the small rank case, where $r < r_1 = r_1(\mathcal{L})$. Denote

$$Q_n^{\circ}(\mathcal{L}, s) = \frac{1}{n^s} \sum_{r<r_1} \sum_q r_n\big(H_r(q)\big). \tag{13}$$

We can use Lübeck's data ([13]) to compute the exact value of $Q_n^{\circ}(\mathcal{L}, s)$ for $n \leq 10^7$.

When $n > 10^7$, we deal with small and large values of q separately. Letting $q_1 = 49$, we can compute the exact values of

$$R_n^1(\mathcal{L}, s) = \frac{1}{n^s} \sum_{r<r_1} \sum_{q<q_1} r_n\big(H_r(q)\big) \tag{14}$$

for all n (they become zero for large enough n). On the other hand, by Lemma 7, we know that

$$\frac{1}{n^s} \sum_{r<r_1} \sum_{q \geq q_1} r_n\big(H_r(q)\big) \leq R_n^2(\mathcal{L}, s),$$

where

$$R_n^2(\mathcal{L}, s) = \frac{1}{n^s} \sum_{\nu;\, N_\nu \leq n} h_\nu\big((n/a)^{1/d} + \varepsilon\big) \tag{15}$$

with $\nu = (a, d, \varepsilon)$ parametrising classes of Lusztig polynomials.

Now[2], Q_n° is bounded from above by $R_n^1 + R_n^2$, and for large enough n, by R_n^2 alone. To resolve the ultimate behaviour of Q_n°, we use still another

[1] In this section and the next, sums over r and q are to be taken over positive integers, starting from r_0, and prime powers, respectively.

[2] Henceforth, we often simplify the notation by suppressing the parameters $\mathcal{L}$ and s.

upper bound for R_n^2:

$$\bar{R}_n^2(\mathcal{L}, s) = \frac{1}{n^s} \sum_\nu h_\nu\big((n/a)^{1/d} + \varepsilon\big). \tag{16}$$

Consider a fixed $\nu = (a, d, \varepsilon)$. Writing m for the degree of the polynomial h_ν, we see that in each summand above, the greatest power of n is m/d. From Lübeck's data, it is quickly verified that $m/d \leq s_{\mathcal{L}} \leq s$. As every h_ν has only non-negative coefficients, this implies that $\bar{R}_n^2$ is decreasing in n. Therefore, $\bar{R}_n^2$ has a limit, and this limit is 0 if $s > s_{\mathcal{L}}$.

For large ranks, we use minimal degree polynomials. We shall deal separately with degrees below and above the gap bound. First, for a universal covering group H of type $\mathcal{L}$, let $r_n^<(H)$ denote the number of irreducible representations of degree n below the gap bound. Then define

$$Q_n^<(\mathcal{L}, s) = \frac{1}{n^s} \sum_{r \geq r_1} \sum_q r_n^<\big(H_r(q)\big). \tag{17}$$

For $r \geq r_1$, the smallest value of the gap bound is $\Gamma(r_1, q_0(\mu))$. Call this integer n_0. Using the minimal degree polynomials, we can compute the exact values of $Q_n^<$ for $n < n_0$.

When $n \geq n_0$, Lemma 8 gives the following upper bound to $Q_n^<$:

$$R_n^<(\mathcal{L}, s) = \frac{1}{n^s} \sum_\nu \sum_{\substack{r \geq r_1 \\ N_\nu(r) \leq n}} g_\nu\big(q_\nu(r, n)\big). \tag{18}$$

Here, $\nu = (a, \varepsilon, \alpha, \beta)$ parametrises classes of minimal degree polynomials. The values of $R_n^<$ can be computed exactly for arbitrarily large n. To bound the eventual behaviour of $R_n^<$, we use the fact that $N_\nu(r) > n$ when $r > r_\nu(n)$. This leads to the following upper bound to $R_n^<(\mathcal{L}, s)$:

$$\bar{R}_n^<(\mathcal{L}, s) = \sum_\nu \frac{1}{n^s}(r_\nu(n) - r_1 + 1) \cdot g_\nu\big(q_\nu(r_1, n)\big). \tag{19}$$

Note that since the g_ν are linear polynomials, each summand in the above expression has the form

$$\frac{1}{n^s}(\log_b n - A)\left(Bn^{\frac{1}{\alpha r_1 + \gamma}} + C\right)$$

with B and C non-negative and $\alpha r_1 + \gamma > 1/s \geq 1/s_{\mathcal{L}}$. This expression becomes positive and decreasing when $n \geq b^A \exp\big((s - \frac{1}{\alpha r_1 + \gamma})^{-1}\big)$, and tends to 0. This means that $\bar{R}_n^<$ is eventually strictly decreasing.

Lastly, let $r_n^>(H)$ stand for the number of irreducible representations of H with degree n above the gap bound. We mean to bound

$$Q_n^>(\mathcal{L}, s) = \frac{1}{n^s} \sum_{r \geq r_1} \sum_q r_n^>\big(H_r(q)\big). \tag{20}$$

This is achieved by using the Fulman–Guralnick bounds for conjugacy class numbers. According to these bounds, the number of classes of $H_r(q)$ is at most $q^r + B_{\mathcal{L}} q^{r-1}$, with $B_{\mathcal{L}}$ given in Table 19. We write

$$R_n^{>}(\mathcal{L}, s) = \frac{1}{n^s} \sum_{r \geq r_1} \left(\sum_{q;\,\Gamma(r,q) \leq n} q^r + B_{\mathcal{L}} q^{r-1} \right), \tag{21}$$

where Γ is the gap bound. This is clearly an upper bound to $Q_n^{>}$.

The exact values of $R_n^{>}$ can be computed for arbitrarily large n. By Lemma 10, we have $\sum_q \left(q^r + B_{\mathcal{L}} q^{r-1}\right) < \Lambda_r(n)$, where

$$\Lambda_r(n) = \frac{1}{2(r+1)} q_\mu(r,n)^{r+1} + \left(\frac{2^{r+1}-1}{2^r-1} + \frac{B_{\mathcal{L}}}{2r} \right) q_\mu(r,n)^r + \frac{(2^r-1)B_{\mathcal{L}}}{2^{r-1}-1} q_\mu(r,n)^{r-1}.$$

Here, $q_\mu(r,n)$ is the upper bound for q presented in (10). It has the form $(an)^{1/(br-c)}$ for some positive integers b and c, with $a = 2$ if $\mathcal{L} = C$ and $a = 1$ otherwise (see Table 18). Since $b \leq c$ for all classical families, we find every term in $\Lambda_r(n)$ decreasing in r (for any fixed n). This enables us to bound $R_n^{>}$ from above simply by

$$\bar{R}_n^{>}(\mathcal{L}, s) = \frac{r_\mu(n) - r_1 + 1}{n^s} \Lambda_{r_1}(n). \tag{22}$$

Here we also used the fact that $\Gamma_\mu(r,q) > n$ when $r > r_\mu(n)$ and $q > q_\mu(r,n)$.

Note that $r_\mu(n)$ has the form $a \log_b n + c$, and the highest power of n in $\Lambda_{r_1}(n)$ is less than s, except when $\mathcal{L} = {}^2A$ and $s = 2/3$. In all the other situations, we conclude that $\bar{R}_n^{>}$ will eventually become decreasing. For the one exceptional case, we note that

$$f_\mu(r,q) = (q-1)q^{2r-3} \geq 2^{-1/2} q^{2r-5/2}.$$

From this inequality, we obtain $q_\mu^*(r,n) = (\sqrt{2}n)^{1/(2r-5/2)}$ to be used in place of $q_\mu(r,n)$ as an upper bound to q. This makes $\bar{R}_n^{>}$ eventually decreasing.

Remark. For $\bar{R}_n^{>}$ to become eventually decreasing, we need the highest power of n in $\Lambda_{r_1}(n)$ to be less than s. This is why we cannot always choose $s_{\mathcal{L}} = 2/h_{\mathcal{L}}$, which would otherwise be optimal. If we would have complete lists of Lusztig polynomials available for larger ranks, we could take $r_1(\mathcal{L})$ to be bigger, and this would let us make $s_{\mathcal{L}}$ smaller.

We are now in a position to prove the claims of the theorem case by case for each classical family.

Case $\mathcal{L} = A'$. Here, s is either 1 or 2/3, and $r_1 = 9$. Set $n_0 = \Gamma(r_1, 2) = 173228$. For $n < n_0$, we compute the exact values of $Q_n^{\circ} + Q_n^{<}$. Also, the exceptional covers $A_2(2)$, $A_2(4)$ and $A_3(2)$ have all representation degrees below n_0. Using the data given in the ATLAS of Finite Groups ([2]), we can compute $Q_n^e(s) = \sum_H r_n(H)/n^s$ for the exceptional covers H. Now, we have $Q_n = Q_n^{\circ} + Q_n^{<} + Q_n^e$ when $n < n_0$, and the maximum of $Q_n(s)$ for $n < n_0$ equals 7/8 for $s = 1$, and 7/4 for $s = 2/3$ (the maximum is obtained at $n = 8$ in both cases). For ease of reference, these maxima are listed below the proof in Table 20 on page 29.

When $n \geq n_0$, we use the upper bounds derived above. Values of R_n^1 and R_n^2 are handled by the computer. There is only one class of minimal degree polynomials, so

$$R_n^<(s) = \sum_{r=9}^{\infty}{}' \frac{1}{n^{s-1/r}} \quad \text{and} \quad \bar{R}_n^<(s) = (\log_2 n - 8)\frac{1}{n^{s-1/9}}.$$

In the dashed ($'$) sum we add the restriction $n \geq N_\nu(r) = 2^{r+1} - 2$.

On the other hand, above the gap bound we have

$$R_n^>(s) = \frac{1}{n^s}\sum_{r=9}^{\infty}\sum_q{}'(q^r + Bq^{r-1}),$$

where $B = 3$, and where we add the restriction $n \geq \Gamma(r,q) = \frac{(q^{r+1}-1)(q^r-q^2)}{(q-1)(q^2-1)}$. Moreover,

$$\bar{R}_n^>(s) = \left(\frac{1}{2}\log_2 n - 7\right)\left(\frac{1}{20n^{s-5/8}} + \frac{6649}{3066n^{s-9/16}} + \frac{511}{85n^{s-1/2}}\right).$$

Set $n_1 = 10^7$ and $n_2 = 2 \cdot 10^{60}$. We define the function F_s as follows:

$$F_s(n) = \begin{cases} Q_n^\circ(s) + R_n^<(s) + R_n^>(s) & \text{when } n_0 \leq n \leq n_1 \\ R_n^1(s) + R_n^2(s) + R_n^<(s) + R_n^>(s) & \text{when } n_1 < n < n_2 \\ \bar{R}_n^2(s) + \bar{R}_n^<(s) + \bar{R}_n^>(s) & \text{when } n \geq n_2. \end{cases}$$

From the earlier discussion, it is clear that $F_s(n)$ is an upper bound to $Q_n(s)$ for all $n \geq n_0$. (For $n \geq n_2$, we have $R_n^1 = 0$.) Direct computation shows that the maximum of $F_s(n)$ for $n_0 \leq n < n_2$ is less than 0.007517 when $s = 1$ and 1.17409 when $s = 2/3$. On the other hand, $F_1(n_2) < 10^{-21}$ and $F_{2/3}(n_2) < 0.1$, and when $n \geq n_2$, some simple calculus tells us that $F_s(n)$ is descending for both values of s.

Let $c_{A',s}$ be the maximal value of $Q_n(s)$ for $n < n_0$ obtained above. As $c_{A',s}$ is greater than the maximum of $F_s(n)$ for $n \geq n_0$ for both values of s, we can conclude that it bounds $Q_n(s)$ from above for all n. This means that $s_n(A') \leq c_{A',s}n^s$, which proves the claim of the theorem in the case $\mathcal{L} = A'$.

Case $\mathcal{L} = {}^2A$. Again, s is either 1 or 2/3, and $r_1 = 9$. Using the same notation as in the previous case, we set $n_0 = \Gamma(r_1, 2) = 57970$, $n_1 = 10^7$ and $n_2 = 2 \cdot 10^{60}$. For $n < n_0$, we compute the exact values of $Q_n^\circ + Q_n^<$. The exceptional covers ${}^2A_3(2)$, ${}^2A_3(3)$ and ${}^2A_5(2)$ have all their representation degrees below n_0. With the help of the Atlas, we can compute $Q_n^e(s) = \sum_H r_n(H)/n^s$ for the exceptional covers H. Now, $Q_n = Q_n^\circ + Q_n^< + Q_n^e$ when $n < n_0$, and the maximum of $Q_n(s)$ in this range equals 2/3 for $s = 1$ (at $n = 6$), and $13/21^{2/3} \approx 1.7080$ for $s = 2/3$ (at $n = 21$). These maxima are listed in Table 20 below the proof.

Suppose then that $n \geq n_0$. Values of R_n^1 and R_n^2 are handled by the computer. We have

$$R_n^<(s) = \sum_{r=9}^{\infty}{}' \frac{1}{n^{s-1/(r-1)}} \quad \text{and} \quad \bar{R}_n^<(s) = (\log_2 n - 7)\frac{1}{n^{s-1/8}}.$$

In the dashed ($'$) sum we add the restriction $n \geq N_\nu(r) = (2^{r+1} - \gcd(r,2))/3$.

On the other hand, above the gap bound we have $R_n^>$ similar to the one in the previous case, with $B = 15$, and if $s = 1$, we get

$$\bar{R}_n^>(s) = \frac{1}{2}(\log_2 n - 13)\left(\frac{1}{20n^{s-2/3}} + \frac{8693}{3066n^{s-3/5}} + \frac{511}{17n^{s-8/15}}\right).$$

For $s = 2/3$, this expression is not eventually decreasing, so we need to make a modification to $q_\mu(r,n)$ according to the discussion below equation (22). This leads to

$$\bar{R}_n^>(2/3) = \frac{1}{2}(\log_2 n - 13)\left(\frac{2^{10/31}}{20n^{2/93}} + \frac{8693 \cdot 2^{9/31}}{3066n^{8/93}} + \frac{511 \cdot 2^{8/31}}{17n^{14/93}}\right).$$

The function F_s is defined as follows:

$$F_s(n) = \begin{cases} Q_n^\circ(s) + R_n^<(s) + R_n^>(s) & \text{when } n_0 \le n \le n_1 \\ R_n^1(s) + R_n^2(s) + R_n^<(s) + R_n^>(s) & \text{when } n_1 < n < n_2 \\ \bar{R}_n^2(s) + \bar{R}_n^<(s) + \bar{R}_n^>(s) & \text{when } n \ge n_2. \end{cases}$$

Now, $F_s(n)$ is an upper bound to $Q_n(s)$ for all $n \ge n_0$. (For $n \ge n_2$, we have $R_n^1 = 0$.) Computation shows that the maximum of $F_s(n)$ for $n_0 \le n < n_2$ is less than 0.075455 when $s = 1$ and 3.46356 when $s = 2/3$. On the other hand, $F_1(n_2) < 10^{-19}$ and $F_{2/3}(n_2) < 0.3$, and when $n \ge n_2$, it can be seen that $F_s(n)$ is descending for both values of s.

Let $c_{^2A,1}$ be the maximum of $Q_n(1)$ for $n < n_0$ obtained above, and let $c_{^2A,2/3}$ be the maximum of $F_{2/3}(n)$ for $n_0 \le n < n_2$. This way we see that, for both s, the value of $c_{^2A,s}$ is an upper bound to $Q_n(s)$ for all n. This proves the claim in the case $\mathcal{L} = {}^2A$.

Case $\mathcal{L} = B$. Here, s is one of 1, 2/3 and 1/2, and $r_1 = 9$. Minimal rank is 3 and only odd q are taken into account. We set $n_0 = \Gamma(r_1, 3) = 22876792454961$ and $n_2 = 10^{20}$. For $n < n_0$, we compute the exact values of $Q_n^\circ + Q_n^<$. The exceptional cover $B_3(3)$ has all its representation degrees below n_0. We compute the values of $Q_n^e(B,s) = r_n(B_3(3))/n^s$. For $n < n_0$, the maximum of $Q_n(s) = Q_n^\circ(s) + Q_n^<(s) + Q_n^e(s)$ equals 2/27 for $s = 1$ (at $n = 27$), 2/9 for $s = 2/3$ (at $n = 27$), and $2/(3\sqrt{3}) \approx 0.38491$ for $s = 1/2$ (at $n = 27$). These maxima are listed in Table 20 below the proof.

Suppose then that $n \ge n_0$. Values of R_n^1 and R_n^2 are handled by the computer. There are 5 classes of minimal degree polynomials, parametrised by ν. Thus, $R_n^<(s)$ is a sum of five terms of the form $\sum'_r \Psi_\nu(s,n,r)$, where each sum is taken over those r for which $n \ge N_\nu(r)$. The $N_\nu(r)$ can be found in Table 13, and the $\Psi_\nu(s,n,r)$ are listed below:

$$\frac{1}{n^s}, \quad \frac{1}{n^s}, \quad \frac{1}{n^s}, \quad \frac{1}{2(n/2)^{s-1/(2r-2)}}, \quad \frac{1}{2n^{s-1/(2r-1)}}.$$

Furthermore, $\bar{R}_n^<(s)$ is the following sum:

$$(\log_3 n - 14)\frac{1}{2n^s} + (\log_3 n - 14)\frac{1}{2n^s} + (\log_3(2n) - 15)\frac{1}{2n^s} \\ + (\log_3(n/2) - 14)\frac{1}{4(n/2)^{s-1/16}} + (\log_3 n - 15)\frac{1}{4n^{s-1/17}}.$$

Above the gap bound, we have $B = 22$ and

$$\bar{R}_n^>(s) = \left(\frac{1}{4}\log_3 n - 6\right)\left(\frac{1}{20n^{s-5/14}} + \frac{14828}{4599n^{s-9/28}} + \frac{11242}{255n^{s-2/7}}\right).$$

The function F_s is defined as follows:

$$F_s(n) = \begin{cases} R_n^1(s) + R_n^2(s) + R_n^<(s) + R_n^>(s) & \text{when } n_0 \leq n < n_2 \\ \bar{R}^1(s) + \bar{R}_n^2(s) + \bar{R}_n^<(s) + \bar{R}_n^>(s) & \text{when } n \geq n_2. \end{cases}$$

Here, $\bar{R}^1(s)$ is the maximum of $R_n^1(s)$ for $n > n_2$. (Note that $R_n^1(s)$ becomes zero eventually.) For $n \geq n_0$, $F_s(n)$ is an upper bound to $Q_n(s)$. Computation shows that the maximum of $F_s(n)$ for $n_0 \leq n < n_2$ is less than $7.2102 \cdot 10^{-9}$ when $s = 1$, less than $2.0469 \cdot 10^{-4}$ when $s = 2/3$ and less than 0.034486 when $s = 1/2$. On the other hand, $F_1(n_2) < 10^{-11}$, $F_{2/3}(n_2) < 10^{-5}$ and $F_{1/2}(n_2) < 0.02$, and when $n \geq n_2$, we see that $F_s(n)$ is descending for each s.

Let $c_{B,s}$ be the maximum of $Q_n(s)$ for $n < n_0$ obtained above. For each s, the value of $c_{B,s}$ is greater than the maximum of $F_s(n)$ for $n_0 \leq n < n_2$. Thus $c_{B,s}$ is an upper bound to $Q_n(s)$ for all n and s. This proves the claim in the case $\mathcal{L} = B$.

Case $\mathcal{L} = C$. Here, $s \in \{1, 2/3, 1/2\}$ and $r_1 = 9$. We set $n_0 = \Gamma(r_1, 2) = 352283520$ and $n_2 = 10^{20}$. For $n < n_0$, we compute the exact values of $Q_n^\circ + Q_n^<$. The exceptional covers $C_2(2)$ and $C_3(2)$ have all their representation degrees below n_0. We compute $Q_n^e(s) = \sum_H r_n(H)/n^s$ for the exceptional covers H. For $n < n_0$, the maximum of $Q_n(s) = Q_n^\circ(s) + Q_n^<(s) + Q_n^e(s)$ equals $1/2$ for $s = 1$ (at $n = 4$), $6/20^{2/3} \approx 0.81433$ for $s = 2/3$ (at $n = 20$), and $3/\sqrt{5} \approx 1.34165$ for $s = 1/2$ (at $n = 20$). These maxima are listed in Table 20 below the proof.

Suppose then that $n \geq n_0$. Values of R_n^1 and R_n^2 are handled by the computer. There are 9 classes of minimal degree polynomials, and $R_n^<$ and $\bar{R}_n^<$ are obtained as in the previous case. Above the gap bound we have $B = 30$, and

$$\bar{R}_n^>(s) = \left(\frac{1}{4}\log_2 n - 6\right)\left(\frac{2^{10/29}}{20n^{s-10/29}} + \frac{5624 \cdot 2^{9/29}}{1533n^{s-9/29}} + \frac{1022 \cdot 2^{8/29}}{17n^{s-8/29}}\right).$$

The function F_s is defined as follows:

$$F_s(n) = \begin{cases} R_n^1(s) + R_n^2(s) + R_n^<(s) + R_n^>(s) & \text{when } n_0 \leq n < n_2 \\ \bar{R}^1(s) + \bar{R}_n^2(s) + \bar{R}_n^<(s) + \bar{R}_n^>(s) & \text{when } n \geq n_2. \end{cases}$$

As in the previous case, $\bar{R}^1(s)$ is just the maximum of $R_n^1(s)$ for $n \geq n_2$. For all $n \geq n_0$, $F_s(n)$ is an upper bound to $Q_n(s)$. Computation shows that the maximum of $F_s(n)$ for $n_0 \leq n < n_2$ is less than $6.2765 \cdot 10^{-5}$ when $s = 1$, less than 0.044328 when $s = 2/3$ and less than 1.17804 when $s = 1/2$. On the other hand, $F_1(n_2) < 10^{-11}$, $F_{2/3}(n_2) < 10^{-4}$ and $F_{1/2}(n_2) < 0.1$, and when $n \geq n_2$, we see that $F_s(n)$ is descending for each s.

Let $c_{C,s}$ be the maximum of $Q_n(s)$ for $n \leq n_0$ obtained above. For each s, the value of $c_{C,s}$ is greater than the maximum of $F_s(n)$ for $n_0 \leq n < n_2$. Thus $c_{C,s}$ is an upper bound to $Q_n(s)$ for all n and s. This proves the claim in the case $\mathcal{L} = C$.

Case $\mathcal{L} = D$. Here, s is one of 1, 2/3 and 1/2, and $r_1 = 8$. Minimal rank is 4. We set $n_0 = \Gamma(r_1, 2) = 4194305$, $n_1 = 10^7$ and $n_2 = 10^{20}$. For $n < n_0$, we compute the exact values of $Q_n^\circ + Q_n^<$. The exceptional cover $D_4(2)$ has all representation degrees below n_0. We compute the values of $Q_n^e(s) = r_n(D_4(2))/n^s$. For $n < n_0$, the maximum of $Q_n(s) = Q_n^\circ(s) + Q_n^<(s) + Q_n^e(s)$ equals $1/8$ for $s = 1$ (at $n = 8$), $3/35^{2/3} \approx 0.28038$ for $s = 2/3$ (at $n = 35$), and $3/\sqrt{35} \approx 0.5071$ for $s = 1/2$ (at $n = 35$). These maxima are listed in Table 20 below the proof.

Suppose then that $n \geq n_0$. Values of R_n^1 and R_n^2 are handled by the computer. There are 5 classes of minimal degree polynomials, and $R_n^<$ and $\bar{R}_n^<$ are obtained as before. Above the gap bound we have $B = 32$, and

$$\bar{R}_n^>(s) = \frac{1}{4}(\log_2 n - 18)\left(\frac{1}{18n^{s-9/22}} + \frac{1021}{255n^{s-4/11}} + \frac{8160}{127n^{s-7/22}}\right).$$

The function F_s is defined as follows:

$$F_s(n) = \begin{cases} Q_n^\circ(s) + R_n^<(s) + R_n^>(s) & \text{when } n_0 \leq n \leq n_1 \\ R_n^1(s) + R_n^2(s) + R_n^<(s) + R_n^>(s) & \text{when } n_1 < n < n_2 \\ \bar{R}^1(s) + \bar{R}_n^2(s) + \bar{R}_n^<(s) + \bar{R}_n^>(s) & \text{when } n \geq n_2. \end{cases}$$

As before, $\bar{R}^1(s)$ is the maximum of $R_n^1(s)$ for $n \geq n_2$. Now, $F_s(n)$ is an upper bound to $Q_n(s)$ for $n \geq n_0$. Computation shows that the maximum of $F_s(n)$ for $n_0 \leq n < n_2$ is less than 0.0010420 when $s = 1$, less than 0.16805 when $s = 2/3$ and less than 2.13399 when $s = 1/2$. On the other hand, $F_1(n_2) < 10^{-10}$, $F_{2/3}(n_2) < 0.001$ and $F_{1/2}(n_2) < 1$, and when $n \geq n_2$, we see that $F_s(n)$ is descending for each s.

For $s \in \{1, 2/3\}$, let $c_{D,s}$ be the maximum of $Q_n(s)$ for $n < n_0$ obtained above, and let $c_{D,1/2}$ be the maximum of $F_{1/2}(n)$ for $n_0 \leq n < n_2$. This way we see that, for each s, the value of $c_{D,s}$ is an upper bound to $Q_n(s)$ for all n. This proves the claim in the case $\mathcal{L} = D$.

Case $\mathcal{L} = {}^2D$. Here, s is one of 1, 2/3 and 1/2, and $r_1 = 8$. Minimal rank is 4. We set $n_0 = \Gamma(r_1, 2) = 4194305$, $n_1 = 10^7$ and $n_2 = 2 \cdot 10^{70}$. For $n < n_0$, we compute the exact values of $Q_n^\circ + Q_n^<$. There are no exceptional covers. For $n < n_0$, the maximum of $Q_n(s) = Q_n^\circ(s) + Q_n^<(s)$ equals $1/34$ for $s = 1$, $1/34^{2/3} \approx 0.095283$ for $s = 2/3$, and $1/\sqrt{34} \approx 0.1715$ for $s = 1/2$ (all reached at $n = 34$). There maxima are listed in Table 20 below the proof.

Suppose then that $n \geq n_0$. Values of R_n^1 and R_n^2 are handled by the computer. There are 5 classes of minimal degree polynomials, and $R_n^<$ and $\bar{R}_n^<$ are obtained as before. Above the gap bound we have $B = 32$, and $\bar{R}_n^>$ is the same as in the previous case.

The function F_s is defined as follows:

$$F_s(n) = \begin{cases} Q_n^\circ(s) + R_n^<(s) + R_n^>(s) & \text{when } n_0 \leq n \leq n_1 \\ R_n^1(s) + R_n^2(s) + R_n^<(s) + R_n^>(s) & \text{when } n_1 < n < n_2 \\ \bar{R}^1(s) + \bar{R}_n^2(s) + \bar{R}_n^<(s) + \bar{R}_n^>(s) & \text{when } n \geq n_2. \end{cases}$$

As before, $\bar{R}^1(s)$ is the maximum of $R_n^1(s)$ for $n \geq n_2$. Now, $F_s(n)$ is an upper bound to $Q_n(s)$ for $n \geq n_0$. Computation shows that the maximum of $F_s(n)$ for $n_0 \leq n < n_2$ is less than 0.0010420 when $s = 1$, less than 0.16805 when $s = 2/3$ and less than 2.13399 when $s = 1/2$. On the other hand,

$F_1(n_2) < 10^{-10}$, $F_{2/3}(n_2) < 0.001$ and $F_{1/2}(n_2) < 1$, and when $n \geq n_2$, we see that $F_s(n)$ is descending for each s.

Let $c_{^2D,1}$ be the maximum of $Q_n(1)$ for $n < n_0$ obtained above, and for $s \in \{2/3, 1/2\}$, let $c_{^2D,s}$ be the maximum of $F_s(n)$ for $n_0 \leq n < n_2$. This way we see that, for each s, the value of $c_{^2D,s}$ is an upper bound to $Q_n(s)$ for all n. This proves the claim in the final case $\mathcal{L} = {}^2D$. □

s	A'	^2A	B	C	D	^2D
1	7/8	2/3	2/27	1/2	1/8	1/34
2/3	7/4	1.7080	2/9	0.8144	0.2804	0.09529
$s_{\mathcal{L}}$	7/4	1.7080	0.3850	1.3417	0.5071	0.1715

TABLE 20. Maximal values of $Q_n(\mathcal{L}, s)$ for small n. (For details, see the case-by-case analysis in the second half of the proof of Theorem 2 above.)

It still remains to check the case of linear groups of rank one.

Proof of Theorem 1. For fields of size 2 and 3, there are no simple groups of type A_1. Simple groups $\mathrm{SL}_2(4)$ and $\mathrm{PSL}_2(9)$ have exceptional covering groups, but we will not consider $\mathrm{SL}_2(4)$ separately, as it is isomorphic to $\mathrm{PSL}_2(5)$. The character degrees and multiplicities of $A_1(9)$ are listed in the ATLAS, and reproduced below in Table 21.

degree	1	3	4	5	6	8	9	10	12	15
multiplicity	1	4	2	2	6	4	3	3	4	2

TABLE 21. Character degrees of the exceptional covering group $A_1(9)$.

For all other finite fields q, the universal covering group is $\mathrm{SL}_2(q)$. For these groups, the generic formulae for complex character degrees and their multiplicities are known, and we give them in Table 22.

q even		q odd	
degree	multiplicity	degree	multiplicity
$q-1$	$q/2$	$q-1$	$(q-1)/2$
q	1	q	3
$q+1$	$(q-2)/2$	$q+1$	$(q-3)/2$
		$(q-1)/2$	2
		$(q+1)/2$	2

TABLE 22. Non-trivial character degrees of $\mathrm{SL}_2(q)$.

With this information, it is easy to compute $s_n(A_1)$ for n as large as we wish. We find that for $n \leq 12$, the largest value of $s_n(A_1)/n$ is $8/3$, and this is obtained only at $n = 3$ (the characters come from $A_1(5)$, $A_1(7)$ and $A_1(9)$). Similarly, the second largest value $7/3$ is obtained only at $n = 6$, and the third largest $3/2$ only at $n = 4$ and $n = 12$.

The largest character degree of $A_1(9)$ is 15, so after this, all degrees come from the characters of $\mathrm{SL}_2(q)$. By Catalan's conjecture (proved by P. Mihăilescu in [16]), no integer greater than 15 can simultaneously have the forms $q_1 - 1$, q_2 and $q_3 + 1$ with different prime powers q_i. Therefore, the maximal number of representations of degree n is reached when n has the forms $q_1 - 1$, $q_2 + 1$, $(q_3 - 1)/2$ and $(q_4 + 1)/2$. (This happens, for example, when $n = 24$.) Adding together the multiplicities for these degrees results in exactly n characters of degree n.

One can also check that $s_{13}(A_1) = 5$, $s_{14}(A_1) = 9$ and $s_{15}(A_1) = 14$, so that $s_n(A_1) < n$ holds for $n \in \{13, 14, 15\}$. Finally, as $n < 8n/3$, we see that the upper bound for $n \leq 12$ holds globally. This proves Theorem 1, and also completes the proof of Theorem 2. □

5. Proving the main result on exceptional groups

In this section, we will prove Theorem 3. Let $\mathcal{E}$ denote the class of finite quasisimple groups of exceptional Lie type. The generic character degrees of these groups are all covered by Lübeck's data ([13, 15]). We proceed as follows:

1. Compute the exact values of $s_n(\mathcal{E})$ up to $n = 10^7$ using Lübeck's data and the Atlas of Finite Groups.
2. Arrange the Lusztig polynomials of $\mathcal{E}$ into classes parametrised by $\nu = (a, d, \varepsilon)$, as was done in the case of classical groups, and use this parametrisation to compute bounds for $s_n(\mathcal{E})$ when $n > 10^7$.

With classical types, we dealt with each Lie type separately, taking sums over ranks. Now we sum over all exceptional types, the set of which we denote Exc.

As in the classical case, we let $q_1 = 49$ and deal with groups of smaller field size separately. With $q \geq q_1$, the classification of the Lusztig polynomials works out the same way as with classical groups. However, instead of dealing with each type separately, we collect together all degree polynomials. For class ν, the final multiplicity polynomial h_ν will then be the sum of maximal multiplicities from each Lie type, again dropping the negative terms. The procedure is analogous to the one in the classical case if we imagine all exceptional types to have the same Dynkin letter but different ranks.

Note, however, that with the so-called Suzuki and Ree types ${}^2B_2$, ${}^2F_4$ and ${}^2G_2$, the indeterminate q in Lübeck's polynomial data denotes the square root of the field size, and hence is not an integer. To mend this, we recompute the polynomials for these types, using as new indeterminate $\hat{q} = q/\sqrt{2}$ for ${}^2B_2$ and ${}^2F_4$, and $\hat{q} = q/\sqrt{3}$ for ${}^2G_2$. Now $\hat{q}$ always assumes integral values. For example, to compute the character degrees of the group ${}^2B_2(8)$, we need to substitute $\hat{q} = \sqrt{8}/\sqrt{2} = 2$ into the new polynomials.

Using these new polynomials in the classification everything works as before, and we have the following lemma corresponding to Lemma 7.

Lemma 11. *Write $H(q)$ for a regular universal covering group of an exceptional Lie type H. With the notation analogous to Lemma 7, we have*

$$\sum_{H\in \mathrm{Exc}}\sum_{q\geq q_1} r_n\big(H(q)\big) \;\leq\; \sum_{\nu;\, N_\nu\leq n} h_\nu\big((n/a)^{1/d}+\varepsilon\big)$$

for all $n > 1$.

We can now proceed directly to proving the main theorem.

Proof of Theorem 3. Let us first deal with the exceptional covering groups. The character degrees of $F_4(2)$, $G_2(2)$, $G_2(3)$, $G_2(4)$, ${}^2B_2(8)$ and ${}^2F_4(2)$ can be found in the Atlas. On the other hand, the Schur multiplier of the simple group of type ${}^2E_6(2)$ is an abelian group of type $2^2\cdot 3$. Every character of the universal covering group has a central subgroup of order 2 in its kernel, and all three factor groups are isomorphic via an outer automorphism. (See [15, Chapter 3.3].) We denote the simple group by ${}^2\breve{E}_6(2)$, and the factor group by $6.{}^2\breve{E}_6(2)$.

From the above, we know that all character degrees of the universal covering group ${}^2E_6(2)$ appear also as degrees of $6.{}^2\breve{E}_6(2)$ (with the same multiplicities). Lübeck has computed the character table of $6.{}^2\breve{E}_6(2)$, and the data is available on his website [14]. From the data we see that the three smallest non-trivial character degrees of $6.{}^2\breve{E}_6(2)$ are 1938, 2432 and 45696, each appearing with multiplicity 1. As the conjugacy class number of $6.{}^2\breve{E}_6(2)$ is 542, we can make the following estimate for $n \geq 45696$:

$$r_n\big({}^2E_6(2)\big) \leq 2\cdot 542 - 3 = 1081.$$

For s in $\{1,\, 2/3,\, 1/2\}$, we are interested in upper bounds of

$$Q_n(s) = \frac{1}{n^s}\sum_{H\in \mathrm{Exc}}\sum_{q} r_n\big(H(q)\big). \tag{23}$$

When $n < n_0 = 45696$, we can use Lübeck's polynomial data together with the Atlas and the discussion above to compute the exact values of $Q_n(s)$. The maximum is found to be 1 for $s = 1$, $7/7^{2/3} \approx 1.91294$ for $s = 2/3$ and $7/7^{1/2} \approx 2.64576$ for $s = 1/2$. (There are three 7-dimensional representations of $G_2(2)$ and four of ${}^2G_2(3)$.)

Let $n_1 = 10^7$ and write $\mathrm{Exc}' = \mathrm{Exc}\setminus\{{}^2E_6\}$. For $n_0 < n \leq n_1$, we can compute the values of

$$\bar{Q}_n(s) = \frac{1}{n^s}\left(\sum_{H\in \mathrm{Exc}'}\sum_{q} r_n\big(H(q)\big) + \sum_{q>2} r_n\big({}^2E_6(q)\big)\right). \tag{24}$$

By the above discussion, $\bar{Q}_n(s) + 1081/n$ is an upper bound to $Q_n(s)$ in this range.

For $n > n_1$, we examine small and large values of q separately. Set $n_2 = 10^{20}$. We can compute the exact values of

$$R^1_n(s) = \frac{1}{n^s}\left(\sum_{H\in \mathrm{Exc}'}\sum_{q<q_1} r_n\big(H(q)\big) + \sum_{2<q<q_1} r_n\big({}^2E_6(q)\big)\right) \tag{25}$$

for all $n < n_2$. We know that $R^1_n(s)$ will eventually become zero, so we also compute

$$\bar{R}^1(s) = \max_{n > n_2} R^1_n(s). \tag{26}$$

For large q, define

$$R^2_n(s) = \frac{1}{n^s} \sum_{\nu;\, N_\nu \le n} h_\nu\big((n/a)^{1/d} + \varepsilon\big), \tag{27}$$

where the notation is as in Lemma 11. By that lemma, we know that

$$\frac{1}{n^s} \sum_{H \in \mathrm{Exc}} \sum_{q \ge q_1} r_n\big(H(q)\big) \le R^2_n(s)$$

for all n. The values of $R^2_n(s)$ can be computed for any specific n. For the ultimate behaviour, we use the following upper bound for $R^2_n(s)$:

$$\bar{R}^2_n(s) = \frac{1}{n^s} \sum_{\nu} h_\nu\big((n/a)^{1/d} + \varepsilon\big). \tag{28}$$

As in the classical case, $\bar{R}^2_n(s)$ is decreasing in n. Namely, the exponent n in each term $h_\nu((n/a)^{1/d} + \varepsilon)$ is at most $1/2$ which is the smallest value of s considered. (As a matter of fact, for some terms coming from groups of type ${}^2B_2$ the exponent is exactly $1/2$, so we could not do better than $s = 1/2$ here.)

Finally, we define the function F as follows:

$$F_s(n) = \begin{cases} \bar{Q}_n(s) + 1081/n & \text{when } n_0 \le n \le n_1 \\ R^1_n(s) + R^2_n(s) + 1081/n & \text{when } n_1 < n < n_2 \\ \bar{R}^1(s) + \bar{R}^2_n(s) + 1081/n & \text{when } n \ge n_2. \end{cases}$$

Now, $F_s(n)$ is an upper bound to $Q_n(s)$ for $n \ge n_0$. Computation shows that the maximum of $F_s(n)$ for $n_0 \le n < n_2$ is less than 0.023657 for $s = 1$, less than 0.84575 for $s = 2/3$ and less than 5.05693 for $s = 1/2$. On the other hand, $F_1(n_2) < 10^{-10}$, $F_{2/3}(n_2) < 0.001$ and $F_{1/2}(n_2) < 1$, and when $n \ge n_2$, we see that $F_s(n)$ is descending for each s.

For $s \in \{1, 2/3\}$, let $c_{\mathcal{E},s}$ be the maximum of $Q_n(s)$ for $n < n_0$ obtained above, and let $c_{\mathcal{E},1/2}$ be the maximum of $F_{1/2}(n)$ for $n_0 \le n < n_2$. Now, for each s, the value of $c_{\mathcal{E},s}$ is an upper bound to $Q_n(s)$ for all n. It follows that $s_n(\mathcal{E}) \le c_{\mathcal{E},s} n^s$, which proves the claim. □

6. Application to Lie groups

In their paper [5], Guralnick, Larsen and Manack prove the following theorem.

Theorem A. *Let G be a simple algebraic group of rank r over an algebraically closed field of characteristic zero. Then the number of conjugacy classes of maximal closed subgroups of G is $O(r)$.*

The proof is based on the classification of closed and connected maximal subgroups of classical Lie groups, originally due to Eugene Dynkin. The classification was restated in the theorem of Martin Liebeck and Gary Seitz, proved in [10], that concerns maximal subgroups of classical algebraic groups over any characteristic. The Liebeck–Seitz theorem is similar to Aschbacher's theorem on finite classical groups. Let K be an algebraically closed field of characteristic $p \geq 0$, and let G be a classical simple algebraic group of rank r (i.e. G is either $\mathrm{SL}_{r+1}(K)$ for $r > 0$, $\mathrm{Sp}_{2r}(K)$ for $r > 1$, $\mathrm{SO}_{2r}(K)$ for $r > 3$, or $\mathrm{SO}_{2r+1}(K)$ for $r > 2$ and $p \neq 2$). Furthermore, let H be a proper closed subgroup of G. The Liebeck–Seitz Theorem states that either H belongs to one of six *geometrical families* of subgroups, denoted $\mathcal{C}_1$–$\mathcal{C}_6$, or to one of two additional families, denoted $\mathcal{S}_1$ and $\mathcal{S}_2$.

Theorem A is now proved as follows. First, Guralnick et al. show that the number of conjugacy classes of subgroups in the geometrical families $\mathcal{C}_1$–$\mathcal{C}_6$ is at most $n + 3 \log n + 3$, where n is the dimension of the defining space of G. Secondly, the family $\mathcal{S}_2$ consists of normalisers of irreducible simple algebraic subgroups, and for this family the authors prove the following: for any $\varepsilon > 0$, the number of conjugacy classes of maximal subgroups in $\mathcal{S}_2$ is $O(n^{\varepsilon})$.

The family $\mathcal{S}_1$, on the other hand, consists of normalisers of irreducible finite quasisimple groups. We saw in Corollary 4 that the number of irreducible characteristic zero representations of finite groups of Lie type is $O(n)$. We have also obtained a similar bound for the alternating groups (see [8]). These bounds are used by Guralnick et al. to bound the number of conjugacy classes in $\mathcal{S}_1$. (At the moment of publication of [5], we had already shown that the number of irreducible representations is $O(n)$, but without a definite constant. This information had been privately communicated to the authors.)

Now, if G is a simple algebraic group of exceptional Lie type, the rank of G is bounded, so there is nothing to prove. Otherwise G is of classical type defined in some space of dimension n, and by the Liebeck–Seitz theorem and the results mentioned above, the total number of conjugacy classes of maximal subgroups is $O(n)$. For each classical type, we have $n \sim r$, which proves Theorem A.

Remark. 1) It should be possible to prove a concrete numerical bound to the number of conjugacy classes of maximal subgroups of complex classical groups. Using the results proved in this paper for embeddings of finite groups of Lie type, as well as similar results for representations of alternating groups obtained by the author, the only unknown that remains is a concrete bound for the number of conjugacy classes in $\mathcal{S}_2$. Using Weyl's formula for character degrees of Lie groups, one could conceivably produce such an estimate.

2) For exceptional groups, the number of conjugacy classes of maximal subgroups remains open, even for complex groups. The isomorphism types of maximal subgroups are known (see e.g. [11]), but the number of embeddings of each type up to conjugation is not.

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